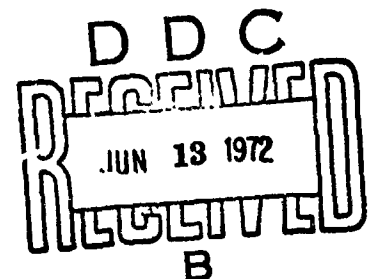AD743066

Some Integrals Involving the Q-Function

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13 ABSTRACT

Some integrals are presented that can be expressed in terms of the Q-function, which is defined as

$$Q(a, b) = \int_b^\infty dx\, x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(ax) \ ,$$

and where I_0 is the modified Bessel function of order zero. Also, integrals of the Q-function are evaluated. Some of the integrals are generalizations of earlier results, but others are new; all derivations are included. Extensions to related integrals are also presented.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Q   Q-function

J_n Bessel function of order n

I_n modified Bessel function of order n

Φ cumulative Gaussian function (see Eq. (64))

Q_{mn} generalization of Q-function (see Eq. (86))

SOME INTEGRALS INVOLVING THE Q-FUNCTION

INTRODUCTION

The performance analysis of phase-incoherent receivers in fading or nonfading media requires evaluating the Q-function. It is defined as [Ref. 1, Eq. (16)]

$$Q(a,b) = \int_b^\infty dx \, x \, \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(ax),$$ (1)

where I_0 is the modified Bessel function of order zero. Physically, the Q-function gives the cumulative distribution function of the envelope of the sum of a sine wave and a narrowband Gaussian process [Ref. 2].

Much past work on performance analysis of receivers in fading media has required evaluating the Q-function or its integrals [Ref. 3-10]. It is the purpose of this report to compile past results and generalize and augment them, where possible. (The method for deriving every relation is included (as an appendix) so that the reader can formulate his own generalizations, where appropriate.) The availability of closed-form expressions for these integrals (in terms of the Q-function) greatly facilitates numerical evaluation, since programs for the Q-function are available [Ref. 11].

Some generalizations and extensions of the Q-function are also included.

GENERAL RELATIONS FOR THE Q-FUNCTION

$$Q(a,0) = 1, \quad Q(a,\infty) = 0 \quad (a < \infty),*$$

$$Q(0,b) = \exp(-b^2/2), \quad Q(\infty,b) = 1 \quad (b < \infty).$$ (2)

---

*Derivations of these and succeeding relations are presented in the appendix.

$$Q(a,b) = \exp\left(-\frac{a^2 + b^2}{2}\right)\sum_{n=0}^{N-1}\left(\frac{a}{b}\right)^n I_n(ab) + a^N \int_b^\infty dx\, x^{-N+1} \exp\left(-\frac{x^2 + a^2}{2}\right)I_N(ax)$$

$$= \exp\left(-\frac{a^2 + b^2}{2}\right)\sum_{n=0}^\infty \left(\frac{a}{b}\right)^n I_n(ab), \quad b \neq 0. \tag{3}$$

$$Q(a,b) = 1 - \exp\left(-\frac{a^2 + b^2}{2}\right)\sum_{n=1}^{N}\left(\frac{b}{a}\right)^n I_n(ab) - a^{-N} \int_0^b dx\, x^{N+1} \exp\left(-\frac{x^2 + a^2}{2}\right)I_N(ax)$$

$$= 1 - \exp\left(-\frac{a^2 + b^2}{2}\right)\sum_{n=1}^\infty \left(\frac{b}{a}\right)^n I_n(ab), \quad a \neq 0. \tag{4}$$

$$Q(a,b) + Q(b,a) = 1 + \exp\left(-\frac{a^2 + b^2}{2}\right)I_0(ab). \tag{5}$$

$$Q(a,a) = \frac{1}{2}\left[1 + \exp(-a^2)\, I_0(a^2)\right]. \tag{6}$$

$$\frac{\partial Q(a,b)}{\partial b} = -b \exp\left(-\frac{a^2 + b^2}{2}\right)I_0(ab). \tag{7}$$

$$\frac{\partial Q(a,b)}{\partial a} = b \exp\left(-\frac{a^2 + b^2}{2}\right)I_1(ab). \tag{8}$$

Note that $Q(a,b)$ is an even function of both a and b (a and b real); this is obvious from (1), (3), or (4). Also, from (1), $Q(a,b)$ is an analytic function of both a and b for all finite complex a and b.

INTEGRALS

In the integrals listed below, the parameters are presumed to be real and positive. However, the results may be generalized to negative or complex parameter values in many cases by symmetry or analytic continuation. For example, (9) holds for all complex a and b and complex p, provided $\mathrm{Re}(p^2) > 0$. This follows because the integrand of (9) is analytic, and the integral is uniformly convergent for $\mathrm{Re}(p^2) > 0$; see Ref. 12, pp. 99-100. The right-hand side of (9) is also analytic for all a and b and for $p \neq 0$. As an example of the use of symmetry, consider (77), which is derived for $b > 0$. Since J_1 is an odd function, we can express $J_1(bx) = \mathrm{sgn}(b) J_1(|b|x)$ for all real b, and utilize the given result by substituting $|b|$ for b and multiplying by $\mathrm{sgn}(b)$. The integrals have been checked numerically.

BESSEL FUNCTIONS, EXPONENTIALS, AND POWERS

$$\int_b^\infty dx\, x \exp(-p^2 x^2/2)\, I_0(ax) = \frac{1}{p^2} \exp\left(\frac{a^2}{2p^2}\right) Q(a/p, bp) \tag{9}$$

$$\int_0^b dx\, x \exp(-p^2 x^2/2)\, I_0(ax) = \frac{1}{p^2} \exp\left(\frac{a^2}{2p^2}\right) [1 - Q(a/p, bp)] \tag{10}$$

$$\int_0^b dx\, x \exp(p^2 x^2/2)\, I_0(ax) = \frac{1}{p^2} \exp\left(-\frac{a^2}{2p^2}\right) [Q(ia/p, ibp) - 1] \tag{11}$$

$$\int_0^b dx\, x \exp(p^2 x^2/2)\, I_0(bp^2 x) = \frac{1}{2p^2}\left[\exp\left(\frac{b^2 p^2}{2}\right) I_0(b^2 p^2) - \exp\left(-\frac{b^2 p^2}{2}\right)\right] \tag{12}$$

$$\int_b^\infty dx \exp(-p^2 x^2/2)\, I_1(ax) = \frac{1}{a} \exp\left(\frac{a^2}{2p^2}\right) [1 - Q(bp, a/p)] \tag{13}$$

$$\int_0^b dx \exp(-p^2 x^2/2)\, I_1(ax) = \frac{1}{a}\left[\exp\left(\frac{a^2}{2p^2}\right) Q(bp, a/p) - 1\right] \tag{14}$$

$$\int_0^b dx \exp(p^2 x^2/2)\, I_1(ax) = \frac{1}{a}\left[\exp\left(-\frac{a^2}{2p^2}\right) Q(ibp, ia/p) - 1\right] \tag{15}$$

$$\int_0^b dx \exp(p^2 x^2/2)\, I_1(bp^2 x) = \frac{1}{bp^2}\left[\frac{1}{2}\exp(-b^2 p^2/2)\right.$$

$$\left. + \frac{1}{2}\exp(b^2 p^2/2)\, I_0(b^2 p^2) - 1\right] \tag{16}$$

$$\int_b^\infty dx\, x^2 \exp(-p^2 x^2/2)\, I_1(ax) = \frac{1}{p^4}\left[a \exp\left(\frac{a^2}{2p^2}\right) Q(a/p, bp)\right.$$

$$\left. + bp^2 \exp(-b^2 p^2/2)\, I_1(ab)\right] \tag{17}$$

$$\int_0^b dx\, x^2 \exp(-p^2 x^2/2)\, I_1(ax) = \frac{1}{p^4}\left[a \exp\left(\frac{a^2}{2p^2}\right)\{1 - Q(a/p, bp)\}\right.$$

$$\left. - bp^2 \exp(-b^2 p^2/2)\, I_1(ab)\right] \tag{18}$$

$$\int_b^\infty dx\, x^3 \exp(-p^2 x^2/2)\, I_0(ax) = \frac{1}{p^6}(a^2 + 2p^2)\exp\left(\frac{a^2}{2p^2}\right) Q(a/p, bp)$$

$$+ \frac{b}{p^4}\exp(-b^2 p^2/2)\left[a\, I_1(ab) + bp^2\, I_0(ab)\right] \tag{19}$$

$$\int_0^b dx\, x^3 \exp(-p^2 x^2/2)\, I_0(ax) = \frac{1}{p^6}(a^2 + 2p^2)\exp\left(\frac{a^2}{2p^2}\right)[1 - Q(a/p, bp)]$$

$$- \frac{b}{p^4}\exp(-b^2 p^2/2)\left[a\, I_1(ab) + bp^2\, I_0(ab)\right] \tag{20}$$

4

$$\int_0^\infty dx \exp(-p^2 x^2/2) \, J_0(ax) \, J_1(bx) = \frac{1}{b} \, [1 - Q(a/p, b/p)] \tag{21}$$

$$\int_0^\infty dx \exp(-p^2 x^2/2) \, I_0(ax) \, I_1(bx) = \frac{1}{b} \, [Q(ia/p, ib/p) - 1] \tag{22}$$

$$\int_0^\infty dx \, \exp(-p^2 x^2/2) \, I_0(ax) \, I_1(ax) = \frac{1}{2a} \left[\exp\!\left(\frac{a^2}{p^2}\right) I_0\!\left(\frac{a^2}{p^2}\right) - 1 \right] . \tag{23}$$

In (24) – (34), $s = \sqrt{p^2 - b^2}$, $u = \sqrt{a(p-s)}$, and $v = \sqrt{a(p+s)}$.

$$\int_a^\infty dx \exp(-px) \, I_0(bx) = \frac{1}{s} \, [2Q(u,v) - \exp(-pa) \, I_0(ab)] , \quad p > b \tag{24}$$

$$\int_0^a dx \exp(-px) \, I_0(bx) = \frac{1}{s} \, [1 + \exp(-pa) \, I_0(ab) - 2Q(u,v)] , \quad p \neq b \tag{25}$$

$$\int_0^a dx \exp(-px) \, I_0(px) = a \exp(-pa) \, [I_0(pa) + I_1(pa)] \tag{26}$$

$$\int_a^\infty dx \exp(-px) \, I_1(bx) = \frac{1}{bs} \, [2pQ(u,v) - (p+s) \exp(-pa) \, I_0(ab)] , \quad p > b \tag{27}$$

$$\int_0^a dx \exp(-px) \, I_1(bx) = \frac{1}{bs} \, [p - s + (p+s) \exp(-pa) \, I_0(ab) - 2pQ(u,v)] , \, p \neq b \tag{28}$$

$$\int_0^a dx \exp(-px) \, I_1(px) = \frac{1}{p} \, [\exp(-pa) \, \{(1+pa) \, I_0(pa) + pa \, I_1(pa)\} - 1] \tag{29}$$

$$\int\limits_{a}^{\infty} dx\, x \exp(-px)\, I_0(bx) = \frac{1}{s^3}\, [\, 2pQ(u,v)$$

$$+ \exp(-pa)\, \{abs\, I_1(ab) + p(as-1)\, I_0(ab)\}\,]\,, \quad p > b \quad (30)$$

$$\int\limits_{0}^{a} dx\, x \exp(-px)\, I_0(bx) = \frac{1}{s^3}\, [\, p - 2pQ(u,v)$$

$$- \exp(-pa)\, \{abs\, I_1(ab) + p(as-1)\, I_0(ab)\}\,]\,, \quad p \neq b \quad (31)$$

$$\int\limits_{0}^{a} dx\, x \exp(-px)\, I_0(px) = \frac{a}{3p} \exp(-pa)\, [\, pa\, I_0(pa) + (pa+1)\, I_1(pa)] \quad (32)$$

$$\int\limits_{a}^{\infty} dx\, x \exp(-px)\, I_1(bx) = \frac{1}{s^3}\, [\, 2bQ(u,v)$$

$$+ \exp(-pa)\, \{aps\, I_1(ab) + b(as-1)\, I_0(ab)\}\,]\,, \quad p > b \quad (33)$$

$$\int\limits_{0}^{a} dx\, x \exp(-px)\, I_1(bx) = \frac{1}{s^3}\, [\, b - 2bQ(u,v)$$

$$- \exp(-pa)\, \{aps\, I_1(ab) + b(as-1)\, I_0(ab)\}\,]\,, \quad p \neq b \quad (34)$$

$$\int\limits_{0}^{a} dx\, x \exp(-px)\, I_1(px) = \frac{a}{3p} \exp(-pa)\, [\, pa\, I_0(pa) + (pa-2)\, I_1(pa)] \quad (35)$$

Q-FUNCTION, EXPONENTIALS, AND x

$$\int\limits_{c}^{\infty} dx\, x \exp(-p^2 x^2/2)\, Q(ax,b) = \frac{1}{p^2}\left[\exp(-p^2 c^2/2)\, Q(ac,b) \right.$$

$$\left. + \exp\left(-\frac{p^2 b^2}{2(a^2+p^2)}\right)\left\{ 1 - Q\left(c\sqrt{a^2+p^2},\ \frac{ab}{\sqrt{a^2+p^2}} \right)\right\}\right] \quad (36)$$

$$\int\limits_{0}^{c} dx\, x\, \exp(-p^2 x^2/2)\, Q(ax,b) = \frac{1}{p^2}\left[\exp\left(-\frac{p^2 b^2}{2(a^2+p^2)}\right) Q\left(c\sqrt{a^2+p^2},\ \frac{ab}{\sqrt{a^2+p^2}}\right) \right.$$

$$\left. - \exp\left(-\frac{p^2 c^2}{2}\right) Q(ac, b) \right] \tag{37}$$

$$\int\limits_{0}^{c} dx\, x\, \exp(p^2 x^2/2)\, Q(ax,b) = \frac{1}{p^2}\left[\exp\left(\frac{p^2 c^2}{2}\right) Q(ac, b) \right.$$

$$\left. - \exp\left(\frac{p^2 b^2}{2(a^2-p^2)}\right) Q\left(c\sqrt{a^2-p^2},\ \frac{ab}{\sqrt{a^2-p^2}}\right) \right], \quad p \neq a \tag{38}$$

$$\int\limits_{0}^{c} dx\, x\, \exp(p^2 x^2/2)\, Q(px, b) = \frac{1}{p^2} \exp\left(\frac{p^2 c^2}{2}\right) [1 - Q(b, pc)] \tag{39}$$

$$\int\limits_{c}^{\infty} dx\, x\, \exp(-p^2 x^2/2)\, Q(b, ax) = \frac{1}{p^2}\left[\exp\left(-\frac{p^2 c^2}{2}\right) Q(b, ac) \right.$$

$$\left. - \frac{a^2}{p^2+a^2} \exp\left(-\frac{b^2 p^2}{2(p^2+a^2)}\right) Q\left(\frac{ab}{\sqrt{p^2+a^2}},\ c\sqrt{p^2+a^2}\right) \right] \tag{40}$$

$$\int\limits_{0}^{c} dx\, x\, \exp(-p^2 x^2/2)\, Q(b, ax) = \frac{1}{p^2}\left[1 - \exp\left(-\frac{p^2 c^2}{2}\right) Q(b, ac) \right.$$

$$\left. - \frac{a^2}{p^2+a^2} \exp\left(-\frac{b^2 p^2}{2(p^2+a^2)}\right)\left\{1 - Q\left(\frac{ab}{\sqrt{p^2+a^2}}, c\sqrt{p^2+a^2}\right)\right\} \right] \tag{41}$$

$$\int_c^\infty dx\, x \exp(p^2 x^2/2)\, Q(b,ax) = \frac{1}{p^2}\left[\frac{a^2}{a^2-p^2}\exp\left(\frac{b^2 p^2}{2(a^2-p^2)}\right) Q\left(\frac{ab}{\sqrt{a^2-p^2}}, c\sqrt{a^2-p^2}\right)\right.$$

$$\left. - \exp\left(\frac{p^2 c^2}{2}\right) Q(b,ac)\right], \quad p < a \qquad (42)$$

$$\int_0^c dx\, x \exp(p^2 x^2/2)\, Q(b,ax) = \frac{1}{p^2}\left[\frac{a^2}{a^2-p^2}\exp\left(\frac{b^2 p^2}{2(a^2-p^2)}\right)\left\{1-Q\left(\frac{ab}{\sqrt{a^2-p^2}}, c\sqrt{a^2-p^2}\right)\right\}\right.$$

$$\left. + \exp\left(\frac{p^2 c^2}{2}\right) Q(b,ac) - 1\right], \quad p \neq a \qquad (43)$$

$$\int_0^c dx\, x \exp(p^2 x^2/2)\, Q(b,px) = \frac{1}{bp^2}[pc \exp(-b^2/2) I_1(bpc) + b \exp(p^2 c^2/2) Q(b,pc) - b] \qquad (44)$$

Q-FUNCTION, BESSEL FUNCTIONS, EXPONENTIALS, AND POWERS

$$\int_0^\infty dx\, x \exp(-p^2 x^2/2)\, I_0(cx)\, Q(ax,b) = \frac{1}{p^2}\exp\left(\frac{c^2}{2p^2}\right) Q\left(\frac{ac}{p\sqrt{p^2+a^2}}, \frac{bp}{\sqrt{p^2+a^2}}\right) \qquad (45)$$

$$\int_0^\infty dx\, x \exp(-p^2 x^2/2)\, I_0(cx)\, Q(b,ax) = \frac{1}{p^2}\left[\exp\left(\frac{c^2}{2p^2}\right) Q\left(\frac{bp}{\sqrt{p^2+a^2}}, \frac{ac}{p\sqrt{p^2+a^2}}\right)\right.$$

$$\left. - \frac{a^2}{p^2+a^2}\exp\left(\frac{c^2-b^2 p^2}{2(p^2+a^2)}\right) I_0\left(\frac{abc}{p^2+a^2}\right)\right] \qquad (46)$$

$$\int_0^\infty dx\, x\, I_0(cx)\, Q(b,ax) = \frac{1}{a^2 c}\exp\left(\frac{c^2}{2a^2}\right)\left[cI_0\left(\frac{bc}{a}\right) + abI_1\left(\frac{bc}{a}\right)\right] \qquad (47)$$

$$\int_0^\infty dx \exp(-p^2 x^2/2)\, I_1(bx)\, Q(c,ax) = \frac{1}{b}\left[\exp\left(\frac{b^2}{2p^2}\right) Q\left(\frac{pc}{\sqrt{p^2+a^2}}, \frac{ab}{p\sqrt{p^2+a^2}}\right) - 1\right] \qquad (48)$$

$$\int_0^\infty dx\ \exp(-p^2x^2/2)\ I_1(bx)\ Q(ax,c) = \frac{1}{b}\left[\exp\left(\frac{b^2}{2p^2}\right)Q\left(\frac{ab}{p\sqrt{p^2+a^2}},\ \frac{pc}{\sqrt{p^2+a^2}}\right)\right.$$

$$\left. - \exp\left(-\frac{c^2}{2}\right)Q\left(\frac{ib}{\sqrt{p^2+a^2}},\ \frac{iac}{\sqrt{p^2+a^2}}\right)\right] \tag{49}$$

$$\int_0^\infty dx\ \exp(-p^2x^2/2)\ I_1(acx)\ Q(ax,c) = \frac{1}{ac}\left[\exp\left(\frac{a^2c^2}{2p^2}\right)Q\left(\frac{a^2c}{p\sqrt{p^2+a^2}},\ \frac{pc}{\sqrt{p^2+a^2}}\right)\right.$$

$$\left. -\frac{1}{2}\exp\left(-\frac{c^2}{2}\frac{p^2-a^2}{p^2+a^2}\right)I_0\left(\frac{a^2c^2}{p^2+a^2}\right) - \frac{1}{2}\exp\left(-\frac{c^2}{2}\right)\right] \tag{50}$$

$$\int_0^\infty dx\ I_1(cx)\ Q(b,ax) = \frac{1}{c}\left[\exp\left(\frac{c^2}{2a^2}\right)I_0\left(\frac{bc}{a}\right) - 1\right] \tag{51}$$

$$\int_0^\infty dx\ x^2\ \exp(-p^2x^2/2)\ I_1(cx)\ Q(ax,b) = \frac{1}{p^4}\left[c\ \exp\left(\frac{c^2}{2p^2}\right)Q\left(\frac{ac}{p\sqrt{p^2+a^2}},\ \frac{bp}{\sqrt{p^2+a^2}}\right)\right.$$

$$\left. +\frac{abp^2}{p^2+a^2}\exp\left(\frac{c^2-b^2p^2}{2(p^2+a^2)}\right)I_1\left(\frac{abc}{p^2+a^2}\right)\right] \tag{52}$$

$$\int_0^\infty dx\ x^2\ \exp(-p^2x^2/2)\ I_1(cx)\ Q(b,ax) = \frac{1}{p^4}\left[c\ \exp\left(\frac{c^2}{2p^2}\right)Q\left(\frac{bp}{\sqrt{p^2+a^2}},\ \frac{ac}{p\sqrt{p^2+a^2}}\right)\right.$$

$$\left. -\frac{a^2}{(p^2+a^2)^2}\exp\left(\frac{c^2-b^2p^2}{2(p^2+a^2)}\right)\left\{c(a^2+2p^2)I_0\left(\frac{abc}{p^2+a^2}\right) + abp^2 I_1\left(\frac{abc}{p^2+a^2}\right)\right\}\right] \tag{53}$$

$$\int_0^\infty dx\ x^2\ I_1(cx)\ Q(b,ax) = \frac{1}{a^4c^2}\exp\left(\frac{c^2}{2a^2}\right)\left[c(c^2+a^2b^2)I_0\left(\frac{bc}{a}\right) + 2ab(c^2-a^2)I_1\left(\frac{bc}{a}\right)\right] \tag{54}$$

Q-FUNCTION OF TWO LINEAR ARGUMENTS, EXPONENTIALS, AND x

In (55) and (56), $s = p^2 + a^2 + b^2$, $t = p^2 + a^2 - b^2$, and $r = \sqrt{s^2 - 4a^2 b^2}$.

$$\int_c^\infty dx\, x\, \exp(-p^2 x^2/2)\, Q(ax, bx) = \frac{1}{p^2}\left[\exp\left(-\frac{p^2 c^2}{2}\right) Q(ac, bc) + \frac{t}{r} Q\left(c\sqrt{\frac{s-r}{2}},\ c\sqrt{\frac{s+r}{2}}\right)\right.$$

$$\left. -\frac{1}{2}\left(1 + \frac{t}{r}\right)\exp\left(-\frac{sc^2}{2}\right) I_0(abc^2)\right] \tag{55}$$

$$\int_0^c dx\, x\, \exp(-p^2 x^2/2)\, Q(ax, bx) = \frac{1}{p^2}\left[\frac{1}{2}\left(1 + \frac{t}{r}\right)\left[1 + \exp\left(-\frac{sc^2}{2}\right) I_0(abc^2)\right]\right.$$

$$\left. - \exp\left(-\frac{p^2 c^2}{2}\right) Q(ac, bc) - \frac{t}{r} Q\left(c\sqrt{\frac{s-r}{2}},\ c\sqrt{\frac{s+r}{2}}\right)\right] \tag{56}$$

$$\int_0^c dx\, x\, Q(ax, bx) = \frac{c^2}{2} Q(ac, bc) + \frac{bc^2}{2(a^2 - b^2)} \exp\left(-\frac{a^2 + b^2}{2} c^2\right)\left[b I_0(abc^2) + a I_1(abc^2)\right]$$

$$- \frac{b^2}{|a^2 - b^2|(a^2 - b^2)}\left[1 + \exp\left(-\frac{a^2 + b^2}{2} c^2\right) I_0(abc^2)\right.$$

$$\left. - 2 Q\left(c \min(a, b),\ c \max(a, b)\right)\right],\quad a \neq b \tag{57}$$

$$\int_0^c dx\, x\, Q(ax, ax) = \frac{c^2}{4}\left[1 + \exp(-a^2 c^2)\left\{I_0(a^2 c^2) + I_1(a^2 c^2)\right\}\right] \tag{58}$$

PRODUCT OF TWO Q-FUNCTIONS

$$\int_0^\infty dx\, x \exp(-p^2 x^2/2)\, Q(ax,b)\, Q(cx,d) = \frac{1}{p^2}\left[\exp\left(-\frac{p^2}{p^2+c^2}\frac{d^2}{2}\right) \right.$$

$$\bullet\, Q\left(\frac{acd}{\sqrt{p^2+c^2}\sqrt{p^2+a^2+c^2}},\ \frac{b\sqrt{p^2+c^2}}{\sqrt{p^2+a^2+c^2}}\right) + \exp\left(-\frac{p^2}{p^2+a^2}\frac{b^2}{2}\right)$$

$$\left.\bullet\left\{1 - Q\left(\frac{d\sqrt{p^2+a^2}}{\sqrt{p^2+a^2+c^2}},\ \frac{abc}{\sqrt{p^2+a^2}\sqrt{p^2+a^2+c^2}}\right)\right\}\right] \tag{59}$$

Q-FUNCTION AND EXPONENTIALS

$$\int_0^\infty dx\, Q(b,ax) = \frac{\sqrt{2\pi}}{4a}\exp(-b^2/4)\left[(b^2+2)\, I_0(b^2/4) + b^2\, I_1(b^2/4)\right] \tag{60}$$

$$\int_0^\infty dx\, [1 - Q(ax,b)] = \frac{\sqrt{2\pi}}{4a}\, b^2\, \exp(-b^2/4)\left[I_0(b^2/4) + I_1(b^2/4)\right] \tag{61}$$

In (62)-(70), $s = \sqrt{a^2+p^2}$.

$$\int_0^\infty dx\, \exp(-p^2 x^2/2)\, Q(b,ax) = \sqrt{\frac{\pi}{2}}\,\frac{1}{p}\left[1 - 2Q\left(\frac{b}{2}\left(1-\frac{p}{s}\right),\frac{b}{2}\left(1+\frac{p}{s}\right)\right)\right.$$

$$\left. + \left(1+\frac{p}{s}\right)\exp\left(-\frac{a^2+2p^2}{s^2}\frac{b^2}{4}\right) I_0\left(\frac{a^2 b^2}{4s^2}\right)\right] \tag{62}$$

$$\int_0^\infty dx\, \exp(-p^2 x^2/2)\, Q(ax,b) = \sqrt{\frac{\pi}{2}}\,\frac{1}{p}\left[2Q\left(\frac{b}{2}\left(1-\frac{p}{s}\right),\frac{b}{2}\left(1+\frac{p}{s}\right)\right)\right.$$

$$\left. - \exp\left(-\frac{a^2+2p^2}{s^2}\frac{b^2}{4}\right) I_0\left(\frac{a^2 b^2}{4s^2}\right)\right] \tag{63}$$

ERROR FUNCTION, BESSEL FUNCTIONS, EXPONENTIALS, AND x

In this report, the error function is defined as

$$\Phi(x) = \int_{-\infty}^{x} dy (2\pi)^{-1/2} \exp(-y^2/2) . \tag{64}$$

$$\int_{0}^{\infty} dx \, x \, \exp(-p^2 x^2/2) \, I_0(bx) \, \Phi(ax) = \frac{1}{p^2} \exp\left(\frac{b^2}{2p^2}\right) \left[1 - Q\left(\frac{b}{2p}\left(1 - \frac{a}{s}\right), \frac{b}{2p}\left(1 + \frac{a}{s}\right)\right) \right]$$

$$+ \frac{1}{2p^2} \left(1 + \frac{a}{s}\right) \exp\left(\frac{b^2}{4s^2}\right) I_0\left(\frac{b^2}{4s^2}\right) \tag{65}$$

$$\int_{0}^{\infty} dx \, x \, I_0(bx) \, \Phi(-ax) = \frac{1}{4a^2} \exp\left(\frac{b^2}{4a^2}\right) \left[I_0\left(\frac{b^2}{4a^2}\right) + I_1\left(\frac{b^2}{4a^2}\right) \right] \tag{66}$$

$$\int_{-\infty}^{\infty} dx \, x \, \exp(-p^2 x^2/2) \, I_0(bx) \, \Phi(ax) = \frac{1}{p^2} \exp\left(\frac{b^2}{2p^2}\right) \left[1 - 2Q\left(\frac{b}{2p}\left(1 - \frac{a}{s}\right), \frac{b}{2p}\left(1 + \frac{a}{s}\right)\right) \right]$$

$$+ \frac{1}{p^2}\left(1 + \frac{a}{s}\right) \exp\left(\frac{b^2}{4s^2}\right) I_0\left(\frac{b^2}{4s^2}\right) \tag{67}$$

$$\int_{0}^{\infty} dx \, \exp(-p^2 x^2/2) \, I_1(bx) \, \Phi(ax) = \frac{1}{b} \exp\left(\frac{b^2}{2p^2}\right) \left[1 - Q\left(\frac{b}{2p}\left(1 - \frac{a}{s}\right), \frac{b}{2p}\left(1 + \frac{a}{s}\right)\right) \right]$$

$$+ \frac{1}{2b}\left[\exp\left(\frac{b^2}{4s^2}\right) I_0\left(\frac{b^2}{4s^2}\right) - 1 \right] \tag{68}$$

$$\int_{0}^{\infty} dx \, I_1(bx) \, \Phi(-ax) = \frac{1}{2b} \left[\exp\left(\frac{b^2}{4a^2}\right) I_0\left(\frac{b^2}{4a^2}\right) - 1 \right] \tag{69}$$

$$\int_{-\infty}^{\infty} dx \exp(-p^2 x^2/2) \, I_1(bx) \, \Phi(ax) = \frac{1}{b} \exp\left(\frac{b^2}{2p^2}\right) \left[1 - 2Q\left(\frac{b}{2p}\left(1 - \frac{a}{s}\right), \frac{b}{2p}\left(1 + \frac{a}{s}\right)\right)\right]$$

$$+ \frac{1}{b} \exp\left(\frac{b^2}{4s^2}\right) I_0\left(\frac{b^2}{4s^2}\right) \tag{70}$$

BESSEL FUNCTIONS OF MORE COMPLICATED ARGUMENTS, EXPONENTIALS, AND x

In (71)-(80), $s = \sqrt{q+c}$, $t = \sqrt{q-c}$, and $u = \sqrt{1-b^2}$.

$$\int_0^{\infty} dx \, x \exp(-qx^2) \, J_0(bx) \, I_0(cx^2) = \frac{1}{2st} \exp\left(-\frac{qb^2}{4(q^2-c^2)}\right) I_0\left(\frac{cb^2}{4(q^2-c^2)}\right), \quad q > c \tag{71}$$

$$\int_0^{\infty} dx \, x \exp(-qx^2) \, J_0(bx) \, I_1(cx^2) = \frac{1}{2c}\left(1 + \frac{q}{st}\right) \exp\left(-\frac{qb^2}{4(q^2-c^2)}\right) I_0\left(\frac{cb^2}{4(q^2-c^2)}\right)$$

$$- \frac{1}{c} Q\left(\frac{b}{2\sqrt{2}}\left(\frac{1}{t} - \frac{1}{s}\right), \frac{b}{2\sqrt{2}}\left(\frac{1}{t} + \frac{1}{s}\right)\right), \quad q > c \tag{72}$$

$$\int_0^{\infty} dx \, x \exp(-cx^2) \, J_0(bx) \, I_1(cx^2) = \frac{1}{\sqrt{2\pi c} \, b} \exp\left(-\frac{b^2}{8c}\right) - \frac{1}{c} \Phi\left(-\frac{b}{2\sqrt{c}}\right) \tag{73}$$

$$\int_0^{\pi} dx \, \frac{1}{\pi} \exp\left(-\frac{a}{1 - b\cos x}\right) = 2Q\left(\frac{1}{u}\sqrt{a(1-u)}, \frac{1}{u}\sqrt{a(1+u)}\right)$$

$$- \exp\left(-\frac{a}{1-b^2}\right) I_0\left(\frac{ab}{1-b^2}\right), \quad b < 1 \tag{74}$$

$$\int_0^{\pi} dx \, \frac{1}{\pi} \exp\left(-\frac{a}{1 - \cos x}\right) = 2 \Phi(-\sqrt{a}) \tag{75}$$

$$\int_0^\infty dx \exp(-qx^2) \, J_1(bx) \, I_0(cx^2) = \frac{1}{b}\left[1 - \exp\left(-\frac{qb^2}{4(q^2-c^2)}\right) I_0\left(\frac{cb^2}{4(q^2-c^2)}\right)\right.$$

$$\left. - 2Q\left(\frac{b}{2st}\sqrt{q-st}, \frac{b}{2st}\sqrt{q+st}\right)\right], \quad q > c \qquad (76)$$

$$\int_0^\infty dx \exp(-cx^2) \, J_1(bx) \, I_0(cx^2) = \frac{1}{b}\left[2\Phi\left(\frac{b}{2\sqrt{c}}\right) - 1\right] \qquad (77)$$

$$\int_0^\pi dx \frac{1}{\pi} \cos x \exp\left(-\frac{a}{1 - b\cos x}\right) = \frac{a}{bu}\left[2uQ\left(\frac{1}{u}\sqrt{a(1-u)}, \frac{1}{u}\sqrt{a(1+u)}\right)\right.$$

$$\left. - \exp\left(-\frac{a}{1-b^2}\right)\left\{(1+u) I_0\left(\frac{ab}{1-b^2}\right) + b I_1\left(\frac{ab}{1-b^2}\right)\right\}\right], \quad b < 1 \qquad (78)$$

$$\int_0^\pi dx \frac{1}{\pi} \cos x \exp\left(-\frac{a}{1 - \cos x}\right) = 2a\left[\Phi(-\sqrt{a}) - (2\pi a)^{-1/2} \exp(-a/2)\right] \qquad (79)$$

$$\int_0^\infty dx \exp(-qx^2) \, J_1(bx) \, I_1(cx^2) = \frac{b}{4csc}\left[\exp\left(-\frac{qb^2}{4(q^2-c^2)}\right)\left\{(q+st) I_0\left(\frac{cb^2}{4(q^2-c^2)}\right)\right.\right.$$

$$\left.\left. + c I_1\left(\frac{cb^2}{4(q^2-c^2)}\right)\right\} - 2stQ\left(\frac{b}{2st}\sqrt{q-st}, \frac{b}{2st}\sqrt{q+st}\right)\right], \quad q > c \qquad (80)$$

$$\int_0^\infty dx \exp(-cx^2) \, J_1(bx) \, I_1(cx^2) = \frac{1}{2c}\left[\left(\frac{2c}{\pi}\right)^{1/2} \exp\left(-\frac{b^2}{8c}\right) - b\,\Phi\left(-\frac{b}{2\sqrt{c}}\right)\right] \qquad (81)$$

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Q-FUNCTION AND x

$$\int_c^\infty dx\ x\ Q(b,ax) = \frac{2+b^2-a^2c^2}{2a^2}\ Q(b,ac) + \frac{c}{2a}\exp\left(-\frac{a^2c^2+b^2}{2}\right)[ac\ I_0(abc)$$

$$+\ b\ I_1(abc)]\tag{82}$$

$$\int_0^c dx\ x\ Q(b,ax) = \frac{2+b^2}{2a^2} - \frac{2+b^2-a^2c^2}{2a^2}\ Q(b,ac) - \frac{c}{2a}\exp\left(-\frac{a^2c^2+b^2}{2}\right)$$

$$\bullet\ [ac\ I_0(abc) + b\ I_1(abc)]\tag{83}$$

$$\int_0^c dx\ x\ Q(ax,b) = \frac{a^2c^2-b^2}{2a^2}\ [1-Q(b,ac)] + \frac{c}{2a}\exp\left(-\frac{a^2c^2+b^2}{2}\right)[ac\ I_0(abc)$$

$$+\ b\ I_1(abc)]\tag{84}$$

$$\int_c^\infty dx\ x\ [1-Q(ax,b)] = \frac{h^2-a^2c^2}{2a^2}\ Q(b,ac) + \frac{c}{2a}\exp\left(-\frac{a^2c^2+b^2}{2}\right)[ac\ I_0(abc)$$

$$+\ b\ I_1(abc)]\tag{85}$$

EXTENSIONS

We define a generalization of the Q-function as

$$Q_{mn}(a,b) = \int_b^\infty dx\ x^m\ \exp\left(-\frac{x^2+a^2}{2}\right)I_n(ax)\ .\tag{86}$$

Then Q_{10} is the standard Q-function. Integration by parts of (86) two different ways yields the relations (see (A-49) and (A-50))

$$Q_{mn}(a,b) = \frac{1}{a}\left[Q_{m+1,n-1}(a,b) - (m+n-1)\, Q_{m-1,n-1}(a,b) \right.$$

$$\left. - b^m \exp\left(-\frac{a^2+b^2}{2}\right) I_{n-1}(ab) \right] \qquad (87)$$

and

$$Q_{mn}(a,b) = aQ_{m-1,n-1}(a,b) + (m-n-1)Q_{m-2,n}(a,b)$$

$$+ b^{m-1} \exp\left(-\frac{a^2+b^2}{2}\right) I_n(ab) . \qquad (88)$$

Equation (87) enables us to relate any three Q_{mn} functions arranged as in pattern A in Fig. 1; (88) accomplishes the relationship depicted by pattern B. That is, knowledge of any two Q_{mn} functions in a pattern enables us to determine the third function.
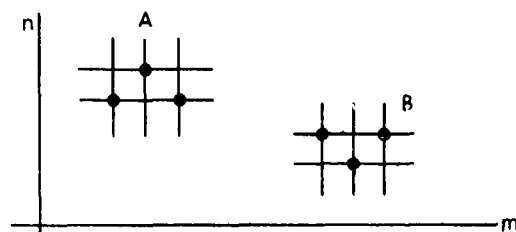
Fig. 1. Interrelated Functions

Substitution of $m = 0$, $n = 1$ in (87) yields

$$Q_{01}(a,b) = \frac{1}{a}\left[Q(a,b) - \exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab) \right] . \qquad (89)$$

(This is equivalent to (13).) Thereby for $m + n$ odd, (87)-(89) enable to evaluating all Q_{mn} functions in terms of a Q-function and Bessel functions.

For $m + n$ even, it appears to be necessary to have two fundamental functions rather than one; these functions could be either Q_{00} and Q_{20}, or

Q_{00} and Q_{11}. Then (87) and (88) enable evaluating all Q_{mn} functions in terms of the two fundamental functions for $m + n$ even. Although it is obvious from (86) that

$$Q_{11}(a,b) = \exp(-a^2/2) \frac{\partial}{\partial a} [\exp(a^2/2) Q_{00}(a,b)] , \qquad (90)$$

we have not been able to reduce the number of fundamental functions below two.

Some examples of integrals that reduce to the fundamental functions are listed below:

$$\int_c^\infty dx\, Q(b, ax) = \frac{1}{a} Q_{20}(b, ac) - c\, Q(b, ac) \qquad (91)$$

$$\int_c^\infty dx\, [1 - Q(ax, b)] = \frac{b}{a} Q_{11}(b, ac) - c\, [1 - Q(ac, b)] \qquad (92)$$

$$\int_c^\infty dx\, I_1(bx)\, \Phi(-ax) = \frac{1}{b} \left[(2\pi)^{-1/2} \exp\left(\frac{b^2}{2a^2}\right) Q_{00}(b/a, ac) - 1_0(bc)\, \Phi(-ac) \right] . \quad (93)$$

Integrals such as

$$\int dx\, x^n \exp(-px)\, I_{0,1}(bx) \qquad (94)$$

can be evaluated by taking derivatives with respect to p of (30)-(34). Also, integrals such as

$$\int dx\, x^{2n+1} \exp(-px^2)\, I_{0,1}(bx^2) \qquad (95)$$

are immediately reduced to the form of (94) by the substitution $x = \sqrt{t}$.

APPLICATION

The error probability fo˙ transmission of M-ary equicorrelated signals over a phase-incoherent Rayleigh fading channel has been considered by Shein [Ref. 9]. If we generalize to the case where, in addition, a threshold must be exceeded [Ref. 5, sect. 6], the probability of correct detection is given by

$$
P_M = \frac{1-\lambda}{1+\beta} \int_{\Gamma/\sqrt{1-\lambda}}^{\infty} du\, u \exp\left(-\frac{1+\beta\lambda}{1+\beta}\frac{u^2}{2}\right) \int_0^{\infty} dv\, v \exp(-v^2/2)
$$

$$
\bullet\; I_0(\sqrt{\lambda}\, uv)\, [1 - Q(\sqrt{\lambda}\, v, u)]^{M-1}. \tag{96}
$$

(This specializes to Shein's result for the threshold equal to zero: $\Gamma = 0$.) For $M = 2$, we use (45) to evaluate the inner integral in (96) and obtain

$$
P_2 = \frac{1-\lambda}{1+\beta} \int_{\Gamma/\sqrt{1-\lambda}}^{\infty} du\, u \exp\left(-\frac{1-\lambda}{1+\beta}\frac{u^2}{2}\right)\left[1 - Q\left(\frac{\lambda u}{\sqrt{1+\lambda}}, \frac{u}{\sqrt{1+\lambda}}\right)\right]. \tag{97}
$$

This integral can be evaluated by utilizing (55) to yield the closed-form solution

$$
P_2 = \exp(-a^2)\, [1 - Q(b\lambda, b)] + \frac{\beta}{r} Q(a\sqrt{c-r},\ a\sqrt{c+r})
$$

$$
+ \frac{1}{2}\left(1 - \frac{\beta}{r}\right)\exp(-a^2 c)\, I_0(b^2\lambda), \tag{98}
$$

where

$$
a = \frac{\Gamma}{\sqrt{2(1+\beta)}}, \quad b = \frac{\Gamma}{\sqrt{1-\lambda^2}}, \quad c = \frac{2 + \beta(1+\lambda^2)}{1-\lambda^2},
$$

$$
r = \left(\beta + \frac{2}{1+\lambda}\right)^{1/2}\left(\beta + \frac{2}{1-\lambda}\right)^{1/2}. \tag{99}
$$

For $\Gamma = 0$, (99) reduces to $(1+\beta/r)/2$, which checks the last equation in Ref. 9. For $M > 2$, integrals of powers of Q are required; in this case, generalizations of (59) are necessary.

Appendix

DERIVATIONS

(2)*: The first three relations in (2) follow immediately from definition (1) and Ref. 13, 6.631 4. The fourth relation results if we express

$$Q(a,b) = 1 - \int_0^b dx\; x\; \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax) \tag{A-1}$$

and use the asymptotic behavior [Ref. 14, 9.7.1]

$$I_0(z) \sim (2\pi z)^{-1/2} \exp(z) \quad \text{as} \quad z \longrightarrow +\infty. \tag{A-2}$$

(3): Integrate by parts repeatedly on (1), with

$$u = I_n(ax)\, x^{-n}, \quad dv = dx\; x\; \exp\left(-\frac{x^2+a^2}{2}\right), \tag{A-3}$$

and employ [Ref. 14, 9.6.28]

$$\frac{d}{dz}\left[z^{-n} I_n(z)\right] = z^{-n} I_{n+1}(z). \tag{A-4}$$

As $N \longrightarrow \infty$ in (3), the integral tends to zero: from Ref. 14, 9.6.18,

$$x^{-N} I_N(ax) = \frac{a^N}{\sqrt{\pi} 2^N\; \Gamma(N+\tfrac{1}{2})} \int_0^\pi d\theta\; \exp(ax\cos\theta)\; (\sin^2\theta)^N$$

$$\leq \frac{a^N \exp(ax)}{\sqrt{\pi} 2^N\; \Gamma(N+\tfrac{1}{2})} \int_0^\pi d\theta\, (\sin^2\theta)^N = \frac{(a/2)^N \exp(ax)}{N!}, \tag{A-5}$$

---

*In this appendix, the number at the beginning of each paragraph refers to the equation of that same number in the main text.

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using Ref. 13, 3.621 3. Then the integral in (3) is upper-bounded by

$$\frac{(a^2/2)^N}{N!} \int_b^\infty dx \; x \; \exp\left(- \frac{(x-a)^2}{2}\right),$$ (A-6)

which tends to zero as $N \longrightarrow \infty$. The infinite series in (3) converges for any $b \neq 0$, as can be seen by the ratio test,

$$\frac{I_{n+1}(ab)}{I_n(ab)} \frac{a}{b} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$ (A-7)

since the ratio of Bessel functions tends to zero as $n \longrightarrow \infty$. (Use Ref. 14, 9.6.26, divide by $I_\nu(z)$, and let $\nu \longrightarrow \infty$.)

(4): Integrate by parts repeatedly on (A-1), with

$$u = \exp\left(-\frac{x^2 + a^2}{2}\right), \quad dv = dx \; x^{n+1} \; I_n(ax),$$ (A-8)

and utilize [Ref. 14, 9.6.28]

$$\frac{d}{dz}\left[z^{n+1} \; I_{n+1}(z)\right] = z^{n+1} \; I_n(z).$$ (A-9)

As $N \longrightarrow \infty$ in (4), the integral tends to zero: when (A-5) is used, the integral in (4) is upper-bounded by

$$\frac{(b^2/2)^N}{N!} \int_0^b dx \; x \; \exp\left(- \frac{(x-a)^2}{2}\right),$$ (A-10)

which tends to zero as $N \longrightarrow \infty$. The infinite series in (4) converges for any $a \neq 0$ by the ratio test (see (A-7) and the text following it).

(5): Interchange a and b in (4), and add to (3).

(6): Let $b = a$ in (5).

(7): Differentiate (1) with respect to b.

(8): Differentiate (5) with respect to a, and use (7) and (A-4).

(9): Substitute $x = pt$ in (1) and reidentify a as a/p, and b as bp.

(10): Use the relation

$$\int_0^b dx\, f(x) + \int_b^\infty dx\, f(x) = \int_0^\infty dx\, f(x) \qquad (A-11)$$

and employ (9) and (2).

(11): Notice first from (1) that $Q(a,b)$ is an analytic function of a and b for all finite a,b. Then replace p by ip in (10), and utilize

$$Q(\pm ia, \pm ib) = Q(ia, ib),\ a,b\ \text{real}, \qquad (A-12)$$

which follows directly from (1) or (3). Also (A-12) is real, as may be seen from (3).

(12): Substitute $a = bp^2$ in (11) and use (6).

(13): Integrate (9) by parts, with

$$u = I_0(ax), \quad dv = dx\, x \exp(-p^2 x^2/2), \qquad (A-13)$$

and then employ (5).

(14): Use (A-11), (13), and (2).

(15): Replace p by ip in (14) and utilize (A-12).

(16): Put $a = bp^2$ in (15) and use (6).

(17): Take the derivative of (9) with respect to a, and employ (8).

(18): Use (A-11), (17), and (2). An extension of (18) is also supplied by replacing p by ip (i.e., p^2 by $-p^2$); the special case of $a = bp^2$ is very simple, upon use of (6).

(8). (19): Take the derivative of (9) with respect to p^2, and employ (7) and

(20): Use (A-11), (19), and (2).

(21): From Ref. 13, 6.633 2, we have

$$b \exp\left(-\frac{a^2 + b^2}{2}\right) I_0(ab) = b \int_0^\infty dt \, t \, \exp(-t^2/2) \, J_0(at) \, J_0(bt)$$

$$= \frac{\partial}{\partial b} \int_0^\infty dt \, \exp(-t^2/2) \, J_0(at) \, b \, J_1(bt) , \quad (A-14)$$

the last step by Ref. 14, 9.1.30. Integrating (A-14) with respect to b, and using (10), yields

$$b \int_0^\infty dt \, \exp(-t^2/2) \, J_0(at) \, J_1(bt) = \int_0^b dx \, x \, \exp\left(-\frac{a^2 + x^2}{2}\right) I_0(ax)$$

$$= 1 - Q(a, b) . \quad (A-15)$$

Let $t = px$ in (A-15) and reidentify a as a/p, and b as b/p.

(22): Replace a by ia, and b by ib in (21), and use [Ref. 14, 9.6.3]

$$J_0(ix) = I_0(x), \quad J_1(ix) = i \, I_1(x) . \quad (A-16)$$

(23): Put b = a in (22) and employ (6).

(24): Using Ref. 14, 9.6.19; Ref. 15, p. 44, 10.203; and defining $q = \sqrt{1 - b^2}$, b < 1, we have

$$\int_a^\infty dt \, \exp(-t) \, I_0(bt) = \int_a^\infty dt \, \exp(-t) \, \frac{1}{\pi} \int_0^\pi d\theta \, \exp(bt \cos \theta)$$

$$= \frac{\exp(-a)}{\pi} \int_0^{\pi} d\theta \; \frac{\exp(ab \cos\theta)}{1 - b \cos\theta}$$

$$= \frac{\exp(-a)}{\pi} \int_0^{\pi} d\theta \; \exp(ab \cos\theta) \frac{1}{q} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{1-q}{b}\right)^n \cos(n\theta) \right]$$

$$= \frac{\exp(-a)}{q} \left[I_0(ab) + 2 \sum_{n=1}^{\infty} \left(\frac{1-q}{b}\right)^n I_n(ab) \right]$$

$$= \frac{1}{q} \left[2Q\left(\sqrt{a(1-q)}, \; \sqrt{a(1+q)}\right) - \exp(-a) \, I_0(ab) \right], \qquad \text{(A-17)}$$

where (4) has been utilized to sum the series. When we let $t = px$ in (A-17) and reidentify a as ap, and b as b/p, (24) follows.

(25): Use (A-11), (24), and (2).

(26): This is a limiting case of (25) as $b \longrightarrow p-$. However, (25) approaches $0/0$ as $b \longrightarrow p-$ (see (6)). Therefore, let $b = \sqrt{p^2-s^2}$ in (25) and apply L'Hospital's rule at $s = 0$. Upon use of (7) and (8), (26) follows.

(27): Integrate by parts, with

$$u = \exp(-px), \quad dv = dx \, I_1(bx), \qquad \text{(A-18)}$$

and then employ (24).

(28): Use (A-11), (27), and (2).

(29): This is a limiting case of (28) as $b \longrightarrow p-$. Apply L'Hospital's rule, as in the derivation for (26), above, and use (7) and (8).

(30): Take the derivative of (24) with respect to p, and utilize (7) and (8).

(31): Use (A-11), (30), and (2).

(32): Define

$$A_n = \int\limits_0^a dx \, x \, \exp(-px) \, I_n(px), \quad n = 0, 1.$$

     (A-19)

In (A-19), for $n = 0$, integrate by parts, with

$$u = \exp(-px), \quad dv = dx \, x \, I_0(px).$$

     (A-20)

Then, using (A-9),

$$A_0 = \frac{a}{p} \exp(-pa) \, I_1(pa) + A_1.$$

     (A-21)

Additionally, in (A-19), for $n = 0$, let instead

$$u = x, \quad dv = dx \, \exp(-px) \, I_0(px).$$

     (A-22)

Then, using (26),

$$A_0 = a^2 \exp(-pa) \, [I_0(pa) + I_1(pa)] - (A_0 + A_1).$$

     (A-23)

Equations (A-21) and (A-23) can now be solved for both A_0 and A_1.

(33): Take the derivative of (27) with respect to p, and utilize (7) and (8).

(34): Use (A-11), (33), and (2).

(35): See the derivation, above, for (32).

(36): Integrate by parts, with

$$u = Q(ax, b), \quad dv = dx \, x \, \exp(-p^2 x^2/2),$$

     (A-24)

and then employ (8) and (13).

(37): Use (A-11), (36), and (2).

(38): Replace p^2 by $-p^2$ in (37).

(39): Integrate by parts, with

$$u = Q(px, b), \quad dv = dx \, x \, \exp(p^2 x^2 / 2), \qquad \text{(A-25)}$$

and then employ (8), (2), (A-4), and (5).

(40): Integrate by parts, with

$$u = Q(b, ax), \quad dv = dx \, x \, \exp(-p^2 x^2 / 2), \qquad \text{(A-26)}$$

and then employ (7) and (9).

(41): Use (A-11), (40), and (2).

(42): Replace p^2 by $-p^2$ in (40). Integral (42) converges for $p < a$ because as $b \longrightarrow \infty$, from (1),

$$Q(a,b) \sim \int_b^{\infty} dx \, x \, \exp\left(-\frac{x^2 + a^2}{2}\right) \frac{\exp(ax)}{\sqrt{2\pi ax}} \sim \sqrt{\frac{b}{a}} \int_b^{\infty} dx (2\pi)^{-1/2} \exp\left(-\frac{(x-a)^2}{2}\right)$$

$$= \sqrt{\frac{b}{a}} \, \Phi(a-b) \sim (2\pi ab)^{-1/2} \exp\left(-\frac{(b-a)^2}{2}\right) \quad \text{as } b \longrightarrow \infty, \qquad \text{(A-27)}$$

where

$$\Phi(t) \equiv \int_{-\infty}^{t} dx (2\pi)^{-1/2} \exp(-x^2 / 2). \qquad \text{(A-28)}$$

(43): Replace p^2 by $-p^2$ in (41).

(44): Integrate by parts, with

$$u = Q(b, px), \quad dv = dx \, x \, \exp(p^2 x^2 / 2), \qquad \text{(A-29)}$$

and then employ (7), (2), and (A-9).

(45): Let f denote the left side of (45). Then, using (7) and Ref. 13, 6.633 2, there follows

$$\frac{\partial f}{\partial b} = - \frac{b}{p^2 + a^2} \exp\left(\frac{-p^2 b^2 + c^2}{2(p^2 + a^2)}\right) I_0\left(\frac{abc}{p^2 + a^2}\right).$$ (A-30)

Since $f = 0$ at $b = \infty$, from (45) and (2), we have

$$f = \int_b^{\infty} dx \frac{x}{p^2 + a^2} \exp\left(\frac{-p^2 x^2 + c^2}{2(p^2 + a^2)}\right) I_0\left(\frac{acx}{p^2 + a^2}\right),$$ (A-31)

and (45) follows upon use of (9).

(46): Employ (5) and then (9); (2); Ref. 13, 6.633 2; and (45) to evaluate the resultant integrals. Next, use (5) again.

(47): Express Q in integral form via (1), interchange integrals, and utilize (A-9) to obtain

$$\frac{1}{c} \int_C^{\infty} dt \, t \, \exp\left(-\frac{t^2 + b^2}{2}\right) I_0(bt) \frac{t}{a} I_1\left(\frac{ct}{a}\right)$$

$$= \frac{1}{c} \frac{\partial}{\partial c} \left[\int_0^{\infty} dt \, t \, \exp\left(-\frac{t^2 + b^2}{2}\right) I_0(bt) I_0\left(\frac{ct}{a}\right)\right]$$

$$= \frac{1}{c} \frac{\partial}{\partial c} \left[\exp\left(\frac{c^2}{2a^2}\right) I_0\left(\frac{bc}{a}\right)\right],$$ (A-32)

where we have employed Ref. 13, 6.633 2. Equation (47) follows directly from (A-32).

(48): Integrate by parts, with

$$u = Q(c, ax), \quad dv = dx \, \exp(-p^2 x^2/2) I_1(bx), \quad v = \frac{1}{b} \exp\left(\frac{b^2}{2p^2}\right) Q(px, b/p),$$ (A-33)

and employ (7), (8), (2), and (45).

(49): Use (5) and then (13), (2), (22), and (48) to evaluate the resultant integrals. Then utilize (5) again.

(50): Let $b = ac$ in (49) and use (6).

(51): Express Q in integral form via (1), interchange integrals, and utilize (A-4) and Ref. 13, 6.633 2.

(52): Take the derivative of (45) with respect to c, and use (A-4) and (8).

(53): Take the derivative of (46) with respect to c, and employ (7) and (A-4).

(54): Take the derivative of (47) with respect to c, and use (A-4) and (A-9).

(55): Integrate by parts, with

$$u = Q(ax, bx), \quad dv = dx\, x\, \exp(-p^2 x^2/2), \qquad (A-34)$$

and employ (2), (7), (8), (24), and (27).

(56): Use (A-11), (55), and (2).

(57): Integrate by parts, with

$$u = Q(ax, bx), \quad dv = dx\, x, \qquad (A-35)$$

and employ (7), (8), and (2). The integral $\int v\, du$ can be reduced to the forms of (31) and (34) by the substitution $y = x^2/2$.

(58): Use (6), let $y = x^2$, and utilize (26).

(59): Express one of the Q-functions in integral form via (1), interchange integrals, and use (45) to evaluate the inner integral. Next, use (36) to evaluate the remaining integral.

(60): Express Q in integral form via (1), and interchange integrals. After taking a derivative with respect to α, utilize Ref. 13, 6.618 4.

(61): Express $1 - Q$ via (A-1), interchange integrals, use Ref. 13, 6.618 4, and (26).

(62): Employ (13) with $p = 1$, to express Q in integral form; interchange integrals; use Ref. 13, 6.631 7; and then (24) and (27).

(63): Use (5); Ref. 13, 6.618 4; and (62).

(64): This is a definition of Φ.

(65): Integrate by parts, with

$$u = \Phi(ax), \quad dv = dx \, x \, \exp(-p^2 x^2/2) \, I_0(bx), \qquad \text{(A-36)}$$

and employ (64), (9), (2), and (62).

(66): Integrate by parts, with

$$u = \Phi(-ax), \quad dv = dx \, x \, I_0(bx), \qquad \text{(A-37)}$$

and employ (A-9) and Ref. 13, 6.618 4, after taking a derivative with respect to β.

(67): Split the integral into a sum of integrals over $(-\infty, 0)$ and $(0, \infty)$. Let $y = -x$ in the integral over $(-\infty, 0)$ and use

$$\Phi(-ax) = 1 - \Phi(ax), \qquad \text{(A-38)}$$

which is deducible from (64). Next, employ (9), (2), and (65).

(68): Integrate by parts, with

$$u = \Phi(ax), \quad dv = dx \, \exp(-p^2 x^2/2) \, I_1(bx), \qquad \text{(A-39)}$$

and employ (64), (13), (2), and (63).

(69): Integrate by parts, with

$$u = \Phi(-ax), \quad dv = dx \, I_1(bx), \qquad \text{(A-40)}$$

and employ (A-4) and Ref. 13, 6.618 4.

(70): Use a procedure similar to that above for (66), and then employ (13), (2), and (69).

(71): Let $x = \sqrt{y}$ and employ Ref. 13, 6.644.

(72): Use Ref. 13, 6.631 7, with $\nu = 1$, to eliminate $I_1(cx^2)$. The resultant integral involving $I_0(cx^2)$ follows from (71). Interchange integrals in the remaining double integral, and use (21) and then (62).

(73): This integral is a limit of (72) as $q \longrightarrow c+$. As $q \longrightarrow c+$, the right side of (72) approaches

$$\frac{1}{2\sqrt{2c}\,t} \exp\left(-\frac{b^2}{8t^2}\right) I_0\left(\frac{b^2}{8t^2}\right) - \frac{1}{c} Q\left(\frac{b}{2\sqrt{2}}\left(\frac{1}{t} - \frac{1}{\sqrt{2c}}\right), \ \frac{b}{2\sqrt{2}}\left(\frac{1}{t} + \frac{1}{\sqrt{2c}}\right)\right), \quad \text{(A-41)}$$

where $t \equiv \sqrt{q-c}$. However, by an approach similar to that given in (A-27), we find that

$$Q(a, b) \sim \sqrt{\frac{b}{a}}\ \Phi(a-b) \quad \text{as} \quad a, b \longrightarrow \infty, \qquad \text{(A-42)}$$

but $b - a \gg 1$ is not required. Equation (73) follows upon use of Ref. 14, 9.7.1.

(74): Define

$$g_n = \frac{1}{\pi} \int_0^\pi dx \ \cos(nx) \ \exp\left(-\frac{a}{1 - b\cos x}\right). \qquad \text{(A-43)}$$

Then

$$\frac{\partial g_0}{\partial a} = -\frac{1}{\sqrt{1-b^2}} \exp\left(-\frac{a}{1-b^2}\right) I_0\left(\frac{ab}{1-b^2}\right), \qquad \text{(A-44)}$$

using Ref. 16, vol. 2, p. 81, Eq. 10. From (A-43), it is seen that $g_0 = 1$ at $a = 0$. When this fact and (25) are utilized, (74) follows.

(75): Take the limit of (74) as $b \longrightarrow 1-$ and use (A-42).

(76): Define

$$f_n' = \int_0^{\infty} dx\ \exp(-qx^2)\ J_1(bx)\ I_n(cx^2).$$ (A-45)

Express I_n in integral form by use of Ref. 14, 9.6.19; interchange integrals; and use Ref. 15, 1.701, to obtain

$$f_n = \frac{1}{\pi b} \int_0^{\pi} d\theta\ \cos(n\theta)\left[1 - \exp\left(-\frac{b^2/4}{q - c\cos\theta}\right)\right].$$ (A-46)

Next, utilize (74).

(77): Take the limit of (76) as $q \longrightarrow c_i+$ and use (A-42) and (A-38).

(78): From (A-43) and Ref. 16, vol. 2, p. 81, Eq. 10,

$$\frac{\partial g_1}{\partial a} = \frac{1}{b\pi} \int_0^{\pi} dx\ \frac{1 - b\cos x - 1}{1 - b\cos x}\ \exp\left(-\frac{a}{1 - b\cos x}\right)$$

$$= \frac{1}{b}\left[g_0 - \frac{1}{\sqrt{1-b^2}}\ \exp\left(-\frac{a}{1-b^2}\right) I_0\left(\frac{ab}{1-b^2}\right)\right].$$ (A-47)

From (A-43), $g_1 = 0$ at $a = 0$. Then, to find g_1 from (A-47), we must be able to evaluate $\int_0^a dx\ g_0$, which, from (A-43) and (74), becomes

$$\int_0^a dx\ g_0 = \int_0^a dx\left[2Q(\alpha\sqrt{x},\ \beta\sqrt{x}) - \exp\left(-\frac{x}{1-b^2}\right) I_0\left(\frac{bx}{1-b^2}\right)\right],$$ (A-48)

where $\alpha = \sqrt{1-u}/u$, $\beta = \sqrt{1+u}/u$, and $u = \sqrt{1-b^2}$. When we let $\sqrt{x} = t$ in the first integral of (A-48) and employ (57) and (25), (78) follows.

(79): Take the limit of (78) as $b \longrightarrow 1-$, and use (A-42) and Ref. 14, 9.7.1.

(80): See the derivation above for (76), set $n = 1$, and employ (78).

(81): Take the limit of (80) as $q \longrightarrow c+$, and use (A-42) and Ref. 14, 9.7.1.

(82): Express Q in integral form via (1), interchange integrals, and use (19) and (1).

(83): Use (A-11), (82), and (2).

(84): Use (5), (83), and (1).

(85): Use (5), (82), and (1).

(87): Litegrate by parts on (86), with

$$u = x^{m+n-1} \exp\left(-\frac{x^2+a^2}{2}\right), \quad dv = dx\, x^{-n+1} I_n(ax), \qquad \text{(A-49)}$$

and employ (A-4) and (86).

(88): Integrate by parts on (86), with

$$u = x^{m-1} I_n(ax), \quad dv = dx\, x \exp\left(-\frac{x^2+a^2}{2}\right), \qquad \text{(A-50)}$$

and employ (A-9) and (86).

(91): Express Q in integral form via (1), interchange integrals, and use (86) and (9).

(92): Express Q in integral form via (13), with p = 1, interchange integrals, and use (86) and (13).

(93): Integrate by parts, with

$$u = \phi(-ax), \quad dv = dx\, I_1(bx), \qquad \text{(A-51)}$$

and employ (A-4) and (86).

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